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# ON $\Lambda(p)$-SUBSETS OF SQUARES

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ABSTRACT

This paper is a follow up of [$B_1$]. It is shown that the sequence of squares $\{n^2 \mid n = 1, 2, \ldots\}$ contains $\Lambda(p)$-subsets of "maximal density", for any given $p > 4$. The proof is based on the probabilistic method developed in [$B_1$] and a precise estimate of the $\Lambda(p)$-constant for the sequence of squares itself. Analogues of this phenomenon are obtained for other arithmetic sets, such as the sequence of $k$th powers $\{n^k \mid n = 1, 2, \ldots\}$ or the sequence of prime numbers. Sections 2 and 3 of the paper are of independent interest to orthogonal system theory.

## 1. Introduction

Following [R], a subset $S$ of the integers $\mathbf{Z}$ is called a $\Lambda(p)$-set ($p > 1$) provided an inequality

$$(1.1) \qquad \left\| \sum_{n \in S} a_n e^{int} \right\|_p \leqq C \left\| \sum_{n \in S} a_n e^{int} \right\|_1$$

holds, for some constant $0 < C < \infty$ and all (finitely supported) scalar sequences $(a_n)_{n \in S}$. Here $\| \quad \|_p$ refers to the $L^p$-norm on the circle $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Using the function space language, the previous condition (1.1) is written as $L^p_S = L^1_S$ (cf. [G–McG]).

A natural problem in this subject is the existence, for given $p \geqq 2$, of $\Lambda(p)$-sets which are not $\Lambda(q)$ for any $q > p$. (The case $p < 2$ is settled by a result stating that for any $S \subset \mathbf{Z}$, $\{ p \in ]1, 2[ \mid S \text{ is } \Lambda_p \}$ is an open interval, see [B–E]).

When $p$ is an even integer $> 2$, the problem may be solved by explicit

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constructions. (The case $p = 4$ goes back to Rudin's paper [R].) Recently the author obtained a probabilistic technique to generate such sets, for any real number $p > 2$ (see [$B_1$]). The existence proof is based on statistical verification among subsets of $\{1, 2, \ldots, N\}$ of a given size. The case $p = 2$ is open so far and cannot be solved by a purely probabilistic extraction argument, in view of the results of [$B_1$].

The search for explicit examples gives the $\Lambda(p)$-set problematic a combinatorial and number theoretic flavor. For instance, an easy condition to ensure that $S$ is a $\Lambda(4)$-set is the following:

$$(1.2) \quad m_1, m_2, n_1, n_2 \in S \quad \text{and} \quad m_1 + m_2 = n_1 + n_2 \Rightarrow \{m_1, m_2\} = \{n_1, n_2\}.$$

(1.2) permits one indeed to construct $\Lambda(4)$-sets which are not $\Lambda(q)$ for any $q > 4$.

Some attention was paid to the sequence of squares $\{n^2\}$. In [R], it was observed that $\{n^2\}$ is not a $\Lambda(4)$-set. The problem whether $\{n^2\}$ is $\Lambda(p)$ for $p < 4$, in particular a $\Lambda(2)$-set, is still open. Let us at this point make a computation of the 4th moment of a polynomial

$$f = \sum_1^N a_n e^{in^2 t}; \tag{1.3}$$

clearly

$$\begin{aligned} \|f\|_4^4 &= \Big\| \sum_{1 \leq m, n \leq N} a_m \bar{a}_n e^{i(m^2 - n^2)t} \Big\|_2^2 \\ &= \sum_{|k| < N^2} \Big| \sum_{\substack{(m-n)(m+n) = k \\ 1 \leq m, n \leq N}} a_m \bar{a}_n \Big|^2 . \end{aligned} \tag{1.4}$$

The number of terms in the inner sum corresponds to the number of divisors of $k$ in the interval $[N - \sqrt{N^2 - k}, \sqrt{k}[$ and hence is bounded by the number $d(k)$ of divisors of $k$.

Invoking the Cauchy–Schwarz inequality, it follows that

$$\|f\|_4^4 \leq \sum_{|k| < N^2} d(k) \sum_{\substack{m^2 - n^2 = k \\ 1 \leq m, n \leq N}} |a_m|^2 |a_n|^2, \tag{1.5}$$

thus

$$\|f\|_4 \leq e^{c \log N / \log \log N} \Big( \sum_1^N |a_n|^2 \Big)^{1/2} . \tag{1.6}$$

On the other hand, if we let $a_n = 1$ $(1 \leqq n \leqq N)$ in (1.3), (1.4) yields that

$$\|f\|_4^4 \geqq \sum_{N^2/4<k<N^2/3} (\#\{N/3<q<N/2 \mid q \mid k\})^2$$

$$(1.7) \qquad \sim \sum_{N/3<q,q'<N/2} N^2 \frac{(q,q')}{q.q'} \qquad (q,q') = \text{greatest common divisor}$$

$$\sim N^2 \log N.$$

Hence $\{n^2\}$ is not a $\Lambda(4)$-set.

In this paper, we will only be concerned with the $\Lambda(p)$-property for $p>2$. If $p>2$, condition (1.1) is equivalent to an inequality

$$(1.8) \qquad \left\| \sum_{n\in S} a_n e^{int} \right\|_p \leqq C\left(\sum |a_n|^2\right)^{1/2},$$

for all coefficient sequences supported by $S$. Denote $K_p(S)$ the smallest constant $C$ satisfying (1.8), i.e. the norm of the identity operator $L_S^2 \to L_S^p$.

Rephrasing (1.6), (1.7), we get thus

$$(1.9) \qquad c(\log N)^{1/4} < K_4(\{n^2 \mid 1 \leqq n \leqq N\}) < \exp\left(c\frac{\log N}{\log\log N}\right).$$

Refining these estimates is an interesting question which, however, will not be pursued here. The upper estimate in (1.9) is used later in proving

PROPOSITION 1.10. *For $p>4$, one has*

$$(1.11) \qquad K_p(\{n^2 \mid n<N\}) \sim N^{1/2-2/p}.$$

Observe that $K_p(S) \geqq CN^{1/2-2/p}$ whenever $S \subset \{1,2,3,\ldots,N^2\}$, $|S|>N$. Indeed

$$K_p(S)^p |S|^{p/2} \geqq \left\| \sum_{n\in S} e^{int} \right\|_p^p \geqq \int_{|t|<1/10N^2} \left| \sum_{n\in S} e^{int} \right|^p$$

$$(1.12) \qquad \geqq CN^{-2}|S|^p.$$

Thus the bound (1.11) is the best one may achieve taking into account the density of the set.

The computation (1.12) also shows that if $S \subset \{1,\ldots,N\}$ has a bounded $\Lambda(p)$-constant, then $|S|<CN^{2/p}$. In [$B_1$] it was shown that in fact the generic subset of $\{1,\ldots,N\}$ of size $\sim N^{2/p}$ has bounded $\Lambda(p)$-constant, for all $2<p<\infty$.

DEFINITION. A $\Lambda(p)$-subset $S$ of $\mathbf{Z}$ has maximal density provided

$$\varliminf_{N=1,2,\ldots} \frac{|S \cap [-N, N]|}{N^{2/p}} > 0. \tag{1.13}$$

Observe that then $S$ cannot be a $\Lambda(q)$-set, for any $q > p$. Rather than invoking (1.12), this fact has also a functional analysis explanation. The linear space $[e^{it}, \ldots, e^{iNt}]$ as subspace of $L^p(\mathrm{T})$ is (uniformly) isomorphic to the $N$-dimensional $l^p$ space, i.e. $l_N^p$ (cf. [Zy]) for which the maximal dimension of Hilbertian subspaces turns out to be $\sim N^{2/p}$ (see [B–D–G–J–N] and [F–L–M] for this and related facts in the context of Dvoretzky's theorem).

Proposition 1.10 is the first step in proving

THEOREM 1. *For all $p > 4$, there are maximal density $\Lambda(p)$-sets contained in the squares.*

This paper is not self-contained. The proof of Theorem 1 uses the techniques of $[B_1]$, in fact a variant of this method, and making the paper self-contained would require a full repetition of $[B_1]$.

The reader may also wish to consult [R] for background material on the $\Lambda(p)$-set problem.

Next, we state some analogues to Theorem 1 for other sets.

THEOEM 2. *For any integer $k \geqq 1$ and $p \geqq p(k)$, there is a maximal density $\Lambda(p)$-set contained in the set $\{n^k\}$.*

Thus $p(1) = 2$, $p(2) = 4$. For $k \geqq 3$ the estimate on $p(k)$ will not be as precise, due to problems similar as in the context of the solution to the Waring problem by the circle method (see [Vaug]).

THEOREM 3. *There are maximal density $\Lambda(p)$-sets contained in the sequence of prime numbers, for any $p > 2$.*

Observe that the prime numbers themselves do not form a $\Lambda(2)$-set, since it may be shown that if $S$ is a $\Lambda(2)$-set then

$$|S \cap [-N, N]| < CN^{\rho}, \tag{1.14}$$

for some constant $C$ and $\rho < 1$.

As above, the letters $c$, $C$ will be used for various constants.

Section 2 deals with a probabilistic result, Section 3 with the consequences, while the last section contains some estimates on exponential sums.

## 2. A probabilistic result

In order to follow this section, the reader will have to consult [$B_1$]. Rather than restricting to characters on a group, we deal with 1-bounded (in $L^\infty$-norm) orthogonal systems on a probability space. Observe that the notion of $\Lambda(p)$-set may be reformulated for more general function systems $\Phi = (\varphi_1, \varphi_2, \ldots)$ and is of interest in orthogonal system theory, namely in connection with Menshov's theorem (see [K–S]). In particular, we introduce the following

DEFINITION. For $p \geqq 2$ and $\Phi$ an orthogonal system, let $K_p(\Phi)$ be the smallest constant $C$ satisfying

$$\left\| \sum a_i \varphi_i \right\|_p \leqq C\left( \sum |a_i|^2 \right)^{1/2}, \tag{2.1}$$

for all finite scalar sequences $(a_i)$.

In the sequel, we will always assume $\Phi$ uniformly bounded by 1.

The main result of this section is the following

PROPOSITION 2.2. *Let $\Phi = (\varphi_1, \ldots, \varphi_n)$, $2 \leqq q < p < 2r$, $r \geqq 2$, $0 < \rho < 1$. There is a subset $S \subset \{1, \ldots, n\}$, $|S| \sim n^\rho$ such that $\Phi_1 = \{\varphi_i \mid i \in S\}$ fulfils the inequality*

$$K_p(\Phi_1) \leqq CK_q(\Phi)^{q/p} n^{1/2(\rho - q/p)} + CK_r(\Phi)^{r/p}. \tag{2.3}$$

*Moreover, this property holds statistically, i.e. for most subsets $S$ of this size. The constant $C$ depends on $p$, $q$, $r$, and $\rho$.*

The proof of Proposition 2.2 is a variant of [$B_1$]. Let us recall some terminology from [$B_1$].

For $1 \leqq m \leqq n$

$$\mathbf{T}_m = \left\{ \bar{a} = (a_i)_{1 \leqq i \leqq n} \;\middle|\; |\bar{a}| \equiv \left( \sum a_i^2 \right)^{1/2} \leqq 1 \text{ and } |\operatorname{supp} \bar{a}| \leqq m \right\}, \tag{2.4}$$

$\delta = n^{-1+\rho}$ and $\{\xi_i(\omega) \mid 1 \leqq i \leqq n\}$ are independent 0,1-valued random variables of expectation $\delta$ (selectors).

Denote further

$$f_{a,\omega} = \sum \xi_i(\omega) a_i \varphi_i, \tag{2.5}$$

$$K_p(\omega) = K_p(\{\varphi_i \mid i \in S_\omega\}), \tag{2.6}$$

where

$$S_\omega = \{1 \leqq i \leqq n \mid \xi_i(\omega) = 1\} \tag{2.7}$$

is the random set.

Denoting $\omega_1, \omega_2, \omega_3$ independent copies of $\omega$, define

$$\begin{aligned} &\bar{K}_{m_1,m_2,m_3}(\omega_1, \omega_2, \omega_3) \\ &= \sup_{|A| \leqq m_1} \sup_{b \in \mathbf{T}_{m_2}} \sup_{c \in \mathbf{T}_{m_3}} \frac{1}{\sqrt{m_1}} \sum_{i \in A} \xi_i(\omega_1) |\langle \varphi_i, f_{b,\omega_2}(1 + |f_{c,\omega_3}|)^{p-2} \rangle|. \end{aligned} \tag{2.8}$$

Denote further $q_0 = \log n$.

Referring to [B$_1$] (more precisely (3.22), (3.23), (3.24), (3.28) in [B$_1$]) one has that

$$\begin{aligned} &\int K_p(\omega)^p d\omega \\ &\leqq C \int K_p(\omega)^{p-1} d\omega \\ &\quad + \int \left\{ \sup_{m_3 < n_0} \sum_{d \geqq 0} [\sup \| \bar{K}_{m_1,m_2,m_3}(\omega_1, \omega_2, \omega_3) \|_{L^{q_0}(d\omega_1)}] \right\} d\omega_2 d\omega_3, \end{aligned} \tag{2.9}$$

where the supremum is taken over all $m_1$, $m_2$ satisfying

$$n_0 \geqq m_1 \geqq 2^d m_2, \qquad m_2 \geqq m_3. \tag{2.10}$$

REMARK. It is easily verified that the other expression (3.24) in [B$_1$] may be estimated by (3.28) of [B$_1$], i.e. (2.9). As in [B$_1$] we evaluate $\| \bar{K}_{m_1,m_2,m_3}(\omega_1, \omega_2, \omega_3) \|_{L^{q_0}(d\omega_1)}$, using lemma 1, lemma 3 of [B$_1$]. These are

LEMMA 2.11. *Let $\mathscr{E}$ be a (bounded) subset of $\mathbf{R}^n_+$, $B = \sup_{x \in \mathscr{E}} |x|$. Let $0 < \delta < 1$ and $(\xi_i)_{i=1}^n$ selectors of mean $\delta$. Let $1 \leqq m \leqq n$. Then*

$$\begin{aligned} &\left\| \sup_{x \in \mathscr{E}, |A| \leqq m} \left[ \sum_{i \in A} \xi_i(\omega) x_i \right] \right\|_{L^{q_0}(d\omega)} \\ &\leqq C \left( \delta m + \frac{q_0}{\log(1/\delta)} \right)^{1/2} B + \left( \log \frac{1}{\delta} \right)^{-1/2} \int_0^B \sqrt{\log N_2(\mathscr{E}, t)}\, dt, \end{aligned} \tag{2.12}$$

*where for $t > 0$, $N_2(\mathscr{E}, t)$ stand for the metrical entropy numbers of $\mathscr{E}$, with respect to the euclidean distance.*

LEMMA 2.13. *Let $\Phi = (\varphi_1, \ldots, \varphi_n)$ be an orthogonal system of functions uniformly bounded by 1, $m \leqq n$ and $2 \leqq s < \infty$. Define*

$$P_m = \left\{ \sum a_i \varphi_i \mid \bar{a} \equiv (a_i) \in \mathbf{T}_m \right\}. \tag{2.14}$$

*Then*

$$\log N_s(P_m, t) \leqq Cm[\log(1 + n/m)]t^{-\nu} \qquad \text{for } t > \tfrac{1}{2}, \tag{2.15}$$

$$\log N_s \quad (P_m, t) \leqq Cm[\log(1 + n/m)]\log \frac{1}{t} \qquad \text{for } 0 < t < \tfrac{1}{2}, \tag{2.16}$$

*where* $C = C_s$ *and* $\nu = \nu(s) > 2$. *Again* $N_s(P, t)$ *stand for the metrical entropy numbers of* $P$ *considered as subset of* $L^s$.

Estimation of $\| \bar{K}_{m_1, m_2, m_3}(\omega_1, \omega_2, \omega_3) \|_{L^{q_0}(d\omega_1)}$ follows next.

Freeze the variables $\omega_2, \omega_3$ for the moment and denote

$$g_{\bar{b}} = f_{\bar{b}, \omega_2}; \qquad h_{\bar{c}} = f_{\bar{c}, \omega_3}. \tag{2.17}$$

Apply (2.11) where $m = m_1$ and

$$\mathscr{E} = \{ (|\langle \varphi_i, g_{\bar{b}}(1 + |h_{\bar{c}}|)^{p-2} \rangle |)_{i=1}^n \mid \bar{b} \in \mathbf{T}_{m_2}, \bar{c} \in \mathbf{T}_{m_3} \}. \tag{2.18}$$

Observe that $q_0/\log(1/\delta) = o(1)$. Hence (2.12) yields

$$\begin{aligned} &\| \bar{K}_{m_1 m_2, m_3}(\omega_1, \omega_2, \omega_3) \|_{L^{q_0}(d\omega_1)} \\ &\leqq C(\delta + m_1^{-1})^{1/2} B + C(m_1 \log n)^{-1/2} \int_0^B \sqrt{\log N_2(\mathscr{E}, t)}\, dt \end{aligned} \tag{2.19}$$

where $B = \sup_{x \in \mathscr{E}} |x|$ is evaluated now. Linearizing and invoking the $L^q, L^{q'}$-duality, Hölder's inequality, one gets

$$\begin{aligned} &\left( \sum |\langle \varphi_i, g_{\bar{b}}(1 + |h_{\bar{c}}|)^{p-2} \rangle |^2 \right)^{1/2} \\ &\leqq K_q(\Phi) \left[ \int |g_{\bar{b}}|^{q'} (1 + |h_{\bar{c}}|)^{(p-2)q'} \right]^{1/q'} \\ &\leqq K_q(\Phi) \| g_{\bar{b}} \|_p \| 1 + |h_{\bar{c}}| \|_p^{p/q' - 1} (1 + \| h_{\bar{c}} \|_\infty)^{p/q - 1}. \end{aligned} \tag{2.20}$$

Remembering (2.17)

$$B \leqq K_q(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/q' - 1} m_3^{1/2(p/q - 1)}. \tag{2.21}$$

On the other hand, using the $L^r, L^{r'}$-duality, also

$$\left( \sum |\langle \varphi_i, g_{\bar{b}}(1 + |h_{\bar{c}}|)^{p-2} \rangle |^2 \right)^{1/2} \leqq K_r(\Phi) \left[ \int |g_{\bar{b}}|^{r'} (1 + |h_{\bar{c}}|)^{(p-2)r'} \right]^{1/r'},$$

$$B \leqq K_r(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/r' - 1} m_3^{1/2(p/r - 1)}. \tag{2.22}$$

We now estimate the euclidean distance between elements of $\mathscr{E}$. Again using $L^r, L^{r'}$-duality

$$
\begin{aligned}
&|(|\langle \varphi_i, g_b(1+|h_c|)^{p-2}\rangle|) - (|\langle \varphi_i, g_{b'}(1+|h_{c'}|)^{p-2}\rangle|)| \\
(2.23)\qquad &\leqq \left(\sum |\langle \varphi_i, g_b(1+|h_c|)^{p-2} - g_{b'}(1+|h_{c'}|)^{p-2}\rangle|^2\right)^{1/2} \\
&\leqq K_r(\Phi)\, \| g_b(1+|h_c|)^{p-2} - g_{b'}(1+|h_{c'}|)^{p-2}\|_{r'}.
\end{aligned}
$$

By the scalar inequality

$$
\begin{aligned}
&|x(1+|y|)^{p-2} - x'(1+|y'|)^{p-2}| \\
(2.24)\qquad &\leqq |x-x'|(1+|y|)^{p-2} + C|x'|(1+|y|) + |y'|)^{(p-3)^+}|y-y'|,
\end{aligned}
$$

where $(p-3)^+ = \max(p-3, 0)$. One gets by Hölder's inequality

$$
\begin{aligned}
(2.25)\qquad & K_r(\Phi)^{-1}\cdot(2.23) \\
&\leqq \| g_b - g_{b'}\|_{p_1} \| 1 + |h_c| \|_p^{p-2} \\
(2.26)\qquad & + C\, \| g_{b'}\|_p \| 1 + |h_c| + |h_{c'}| \|_p^{(p-3)^+} \| h_c - h_{c'}\|_{p_2},
\end{aligned}
$$

where

$$
(2.27)\qquad \frac{r'}{p_1} + \frac{(p-2)r'}{p} = 1, \quad \frac{r'}{p} + \frac{r'(p-3)^+}{p} + \frac{r'}{p_2} = 1.
$$

Observe at this point that the existence of finite $p_1$, $p_2$ fulfilling (2.27) is implied by the hypothesis $p < 2r$.

Again by (2.17)

$$
(2.28)\qquad (2.25) \leqq K_p(\omega_3)^{p-2} \| g_b - g_{b'}\|_{p_1},
$$

$$
(2.29)\qquad (2.26) \leqq CK_p(\omega_2)K_p(\omega_3)^{(p-3)^+} \| h_c - h_{c'}\|_{p_2}.
$$

By (2.23), it follows thus that for $\mathscr{E}$ defined by (2.18)

$$
\begin{aligned}
&\log N_2\left(\mathscr{E}, \frac{t}{K_r(\Phi)}\right) \leqq \log N_{p_1}(P_{m_2}, ctK_p(\omega_3)^{-p+2}) \\
(2.30)\qquad &\qquad + \log N_{p_2}(P_{m_3}, ctK_p(\omega_2)^{-1}K_p(\omega_3)^{-(p-3)^+}).
\end{aligned}
$$

Substituting the bounds (2.21), (2.22) on $B$ and the entropy estimate (2.30), it follows that

$$
\begin{aligned}
(2.19) \leqq\ & C\delta^{1/2} K_q(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/q'-1} m_3^{1/2(p/q-1)} \\
& + C m_1^{-1/2} K_r(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/r'-1} m_3^{1/2(p/r-1)} \\
& + C(m_1 \log n)^{-1/2} K_r(\Phi) K_p(\omega_3)^{p-2} \left( \int_0^\infty \sqrt{\log N_{p_1}(P_{m_2}, t)}\, dt \right) \\
& + K_r(\Phi) K_p(\omega_2) K_p(\omega_3)^{(p-3)^+} \left( \int_0^\infty \sqrt{\log N_{p_2}(P_{m_3}, t)}\, dt \right) \Big]. \qquad (2.31)
\end{aligned}
$$

From (2.15), (2.16) in Lemma 2.13, one gets for $s < \infty$

$$
\int_0^\infty \sqrt{\log N_s(P_m, t)}\, dt \leqq C\sqrt{m \log n}. \qquad (2.32)
$$

We substitute (2.32) in (2.31) and obtain an estimate on

$$
\begin{aligned}
& \| \bar{K}_{m_1,m_2,m_3}(\omega_1, \omega_2, \omega_3) \|_{L^{q}(d\omega_1)} \\
& \leqq C K_q(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/q'-1} \left( \frac{n_0}{n} \right)^{1/2} m_3^{1/2(p/q-1)} \\
& + C K_r(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/r'-1} m_2^{-1/2} m_3^{1/2(p/r-1)} \\
& + C K_r(\Phi) K_p(\omega_3)^{p-2} \left( \frac{m_2}{m_1} \right)^{1/2} + C K_r(\Phi) K_p(\omega_2) K_p(\omega_3)^{(p-3)^+} \left( \frac{m_3}{m_1} \right)^{1/2}. \qquad (2.33)
\end{aligned}
$$

Going back to (2.9) and taking for fixed $d$ the corresponding supremum yields the expression

$$
\begin{aligned}
& C K_q(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/q'-1} \left( \frac{n_0}{n} \right)^{1/2} m_3^{1/2(p/q-1)} \\
& + C K_r(\Phi) K_p(\omega_2) K_p(\omega_3)^{p/r'-1} 2^{-d/2} m_3^{p/2r-1} \\
& + C 2^{-d/2} K_r(\Phi) [K_p(\omega_3)^{p-2} + K_p(\omega_2) K_p(\omega_3)^{(p-3)^+}]. \qquad (2.34)
\end{aligned}
$$

Summation over $d = 0, 1, 2, \ldots$ and taking supremum over $m_3 < n_0$ yields therefore after integration in $\omega_2$, $\omega_3$

$$\begin{aligned}\int K_p(\omega)^p d\omega &\leqq \left[\int K_p(\omega)^p d\omega\right]^{(p-1)/p} \\ &+ CK_q(\Phi)\left[\int K_p(\omega)^p d\omega\right]^{1/q'}\left(\frac{n_0}{n}\right)^{1/2}\sup_{m_3<n_0}\left[m_3^{1/2(p/q-1)}\log\frac{n_0}{m_3}\right] \\ &+ CK_r(\Phi)\left[\int K_p(\omega)^p d\omega\right]^{1/r'} + CK_r(\Phi)\left[\int K_p(\omega)^p d\omega\right]^{((p-3)^+ + 1)/p}. \end{aligned} \tag{2.35}$$

Since $2r > p > q$, it follows easily from (2.35) that

$$\int K_p(\omega)d\omega \leqq CK_q(\Phi)^{q/p} n^{1/2(p-q/p)} + CK_r(\Phi)^{r/p}. \tag{2.36}$$

Since $K_p(\omega)$ is the average $K_p$-bound for subsystems $\Phi_1$ of $\Phi$ of size $\delta n = n^\rho$, Proposition 2.2 is proved.

REMARK. The reader will find the proof of Lemmas 2.11, 2.13 in [B$_1$]. We also refer to [B–L–M] for results related to Lemma 2.13.

## 3. Consequences

In this section, we draw some consequences of Proposition 2.2. Some of them require an iteration of (2.3). We did state (2.3) for generic subsets of the corresponding size on purpose, since sometimes one needs the set to satisfy (2.3) for various parameter choices of $p$, $q$, $r$.

Letting $q < p < 2q$ and $\rho = q/p$ in (2.3), it follows that

$$K_p(\Phi_1) \leqq CK_q(\Phi)^{q/p} \qquad \text{for } |\Phi_1| \sim |\Phi|^{q/p}. \tag{3.1}$$

More generally, if $q < p < \infty$, consider a chain

$$q = p_0 < p_1 < p_2 < \cdots < p_j = p$$

satisfying $p_i < 2p_{i-1}$.

From (3.1) and iteration, a sequence

$$\Phi \supset \Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_j \equiv \Psi$$

is obtained, where

$$|\Phi_i| \sim |\Phi_{i-1}|^{p_{i-1}/p_i}, \tag{3.2}$$

$$K_{p_i}(\Phi_i) \leqq CK_{p_{i-1}}(\Phi_{i-1})^{p_{i-1}/p_i}. \tag{3.3}$$

Hence

$$|\Psi| \sim |\Phi|^{q/p}, \tag{3.4}$$

$$K_p(\Psi) \leqq CK_q(\Phi)^{q/p}. \tag{3.5}$$

For $q = 2$, $K_q(\Phi) = 1$ ($\Phi$ is a 1-bounded ONS). Hence, for all $p > 2$, there is a subset $\Psi$ of $\Phi$, $|\Psi| \sim |\Phi|^{2/p}$ with bounded $K_p$ constant. This is the main result of [$B_1$]. We need the following generalization:

THEOREM 4. *Let $2 \leqq q < p < \infty$ and $\Phi$ be a finite uniformly bounded ONS. Then there exist $\Psi \subset \Phi$ satisfying*

$$K_p(\Psi) < C \quad \textit{and} \quad |\Psi| \sim K_q(\Phi)^{-2q/p} |\Phi|^{q/p}. \tag{3.6}$$

We did state this result as a theorem, since we believe it is of interest to orthogonal system theory, besides the applications in this paper.

Observe that by (3.4), (3.5) it suffices to prove the statement when $q < p < \frac{5}{4}q$. If $p < 4$, one may apply (2.3) with $r = 2$ and obtain the desired result. If $p \geqq 4$, a problem appears when estimating $K_r(\Phi)$ for a suitable value of $r$ and we use again an iteration procedure.

PROOF OF 3.6. Choose $p/2 < r < \frac{3}{4}q$ and consider a sequence

$$2 = r_0 < r_1 < \cdots < r_j = r, \quad r_i < \tfrac{4}{3} r_{i-1}, \quad r_{i-1} < q/2 \quad (i \leqq j). \tag{3.7}$$

Consider also a sequence

$$q = p_0 < p_1 < \cdots < p_j < p < \tfrac{5}{4}q, \quad p_i < p_{i-1} r_i / r_{i-1}. \tag{3.8}$$

Take $t_i$ satisfying

$$p_i/2 < t_i < \tfrac{3}{4} p_i - r_i/4, \tag{3.9}$$

$$0 < \theta_i < 1 \quad \text{satisfying} \; \frac{1}{t_i} = \frac{\theta_i}{q} + \frac{1-\theta_i}{r_{i-1}}. \tag{3.10}$$

Observe that by (3.7), (3.8), $r_{i-1} < t_i < q$.

Define

$$r_i' = r_i \frac{p_{i-1}}{p_i}, \tag{3.11}$$

$$0 < \tau_i < 1 \quad \text{satisfying} \; \frac{1}{r_i'} = \frac{\tau_i}{q} + \frac{1-\tau_i}{r_{i-1}}. \tag{3.12}$$

Observe again that $r_{i-1} < r_i' < q$.

Apply (2.3) with $2 \leqq p_{i-1} < p_i < 2t_i$, $0 < \rho_i < 1$, to the system $\Phi_{i-1}$. Hence most $\Phi_i \subset \Phi_{i-1}$ of size

$$|\Phi_i| = |\Phi_{i-1}|^{\rho_i} \tag{3.13}$$

satisfy

$$K_{p_i}(\Phi_i) \leqq CK_{p_{i-1}}(\Phi_{i-1})^{p_{i-1}/p_i} |\Phi_i|^{1/2} |\Phi_{i-1}|^{-p_{i-1}/2p_i} + CK_{t_i}(\Phi_{i-1})^{t_i/p_i}. \tag{3.14}$$

Since $2 < r_i' < r_i < 2r_{i-1}$, also

$$K_{r_i}(\Phi_i) \leqq CK_{r_i'}(\Phi_{i-1})^{r_i'/r_i} |\Phi_i|^{1/2} |\Phi_{i-1}|^{-r_i'/2r_i} + CK_{r_{i-1}}(\Phi_{i-1})^{r_{i-1}/r_i}. \tag{3.15}$$

Imposing the condition

$$K_{r_i}(\Phi_i) < C \tag{3.16}$$

leads by (3.15) to the condition (on $\rho_i$)

$$|\Phi_i| \sim |\Phi_{i-1}|^{r_i'/r_i} K_{r_i'}(\Phi_{i-1})^{-2r_i'/r_i}. \tag{3.17}$$

We also want the first term in (3.14) to dominate the second, i.e., by (3.17), (3.11)

$$K_{r_i'}(\Phi_{i-1})^{p_{i-1}} K_{t_i}(\Phi_{i-1})^{t_i} \leqq K_{p_{i-1}}(\Phi_{i-1})^{p_{i-1}}. \tag{3.18}$$

By (3.12), (3.16) and interpolation

$$K_{r_i'}(\Phi_{i-1}) \leqq CK_q(\Phi_{i-1})^{\tau_i}. \tag{3.19}$$

Similarly, by (3.10), (3.16)

$$K_{t_i}(\Phi_{i-1}) \leqq CK_q(\Phi_{i-1})^{\theta_i}. \tag{3.20}$$

Thus (3.18) will be fulfilled if

$$\tau_i p_{i-1} + \theta_i t_i \leqq p_{i-1}. \tag{3.21}$$

Substitution of (3.11), (3.12), (3.10) in the left member of (3.21) permits one to derive the inequality from (3.7), (3.8), (3.9).

Since the first member in (3.14) dominates the second, one has

$$K_{p_i}(\Phi_i)^{p_i} |\Phi_{i-1}|^{p_{i-1}/2} \leqq CK_{p_{i-1}}(\Phi_{i-1})^{p_{i-1}} |\Phi_i|^{p_i/2} \quad (1 \leqq i \leqq j) \tag{3.22}$$

from where clearly

$$K_{p_j}(\Phi_j)^{p_j} |\Phi|^{q/2} \leqq CK_q(\Phi)^q |\Phi_j|^{p_j/2}. \tag{3.23}$$

also, by (3.16)

$$K_r(\Phi_j) < C. \tag{3.24}$$

Apply now once more (2.3) with $p_j < p < 2r$, $0 < \rho < 1$ to get

$$\Psi \subset \Phi_j, \quad |\Psi| = |\Phi_j|^\rho$$

such that, by (3.23),

$$K_p(\Psi) \leqq CK_{p_j}(\Phi_j)^{p_j/p} |\Psi|^{1/2} |\Phi_j|^{-p_j/2p}. \tag{3.25}$$

Choosing $\rho$ to achieve boundedness, one gets

$$|\Psi| \sim K_{p_j}(\Phi_j)^{-2p_j/p} |\Phi_j|^{p_j/p}. \tag{3.26}$$

Hence (3.6) holds, by (3.23), (3.26). This completes the proof of Theorem 4. ■

To establish Theorems 1, 2, and 3, the following consequence of Theorem 4 is used:

Corollary 5. *Let $S$ be a subset of $\{1, \ldots, N\}$, $2 < q < \infty$, $|S| > N^{2/q}$. Assume $K_q(S)$ "minimal", i.e.*

$$K_q(S) \leqq C|S|^{1/2} N^{-1/q}. \tag{3.27}$$

*Then, for $q < p$, there is a subset $S_0$ of $s$ satisfying*

$$|S_0| \sim N^{2/p} \quad \text{and} \quad K_p(S_0) < C. \tag{3.28}$$

Proof. Immediate from Theorem 4. ■

In the case of the squares $\{n^2 \mid 1 \leqq n \leqq \sqrt{N}\}$, the minimality of the $\Lambda_q$-constant is shown for all $q > 4$. This is done in Section 4 of this paper.

For each power $k$, one may prove minimality of the $\Lambda_q$-constant of $\{n^k \mid 1 \leqq n \leqq N^{1/k}\}$ for sufficiently large $q$.

For the prime numbers $\{p \mid 1 \leqq p \leqq N\}$, this minimality property holds for all $q > 2$.

The statements in Theorems 1, 2, and 3 then follow immediately from Corollary 5, at least the local version. Passing from the local to the sequence version is done similarly as in [$B_1$], using for instance Littlewood–Paley theory (see the introduction of [$B_1$]).

## 4. Some estimates on exponential sums

In view of the discussion at the end of the previous section, Theorem 1 will follow from the inequality

$$K_p(\{n^2 \mid 1 \leqq n \leqq N\}) \leqq C_p N^{1/2-2/p}, \tag{4.1}$$

for $p > 4$.

Clearly (4.1) follows from a distributional inequality,

$$\sum_1^N |a_n|^2 \leqq 1,$$

$$\operatorname{mes}\left\{t \in \mathbf{T} \;\middle|\; \left|\sum_1^N a_n e^{2\pi i n^2 t}\right| > \delta N^{1/2}\right\} \leqq C_\varepsilon \delta^{-4-\varepsilon} N^{-2}, \tag{4.2}$$

for $0 < \delta < 1$ and where $C_\varepsilon$ may be taken arbitrarily small.

Denoting by $\alpha(N)$ functions of $N$ growing slower than any power of $N$, it follows from (1.9) that

$$\operatorname{mes}\left\{t \in \mathbf{T} \;\middle|\; \left|\sum_1^N a_n e^{2\pi i n^2 t}\right| > \delta N^{1/2}\right\} \leqq \alpha(N) \delta^{-4} N^{-2} \tag{4.3}$$

Thus (4.3) implies (4.2), except for large $\delta$, i.e. $\delta > 1/\alpha(N)$.

To take care of such $\delta$, we use the explicit description of the exponential sum

$$f(t) \equiv \sum_1^N e^{2\pi i n^2 t} \tag{4.4}$$

on major arcs. Thus letting $\nu = 100^{-1}$, define for $1 \leqq a \leqq q \leqq N^\nu$, $(a, q) = 1$

$$\mathcal{M}(q, a) = \{t \in [0, 1] \mid |t - a/q| \leqq N^{\nu-2}\}, \tag{4.5}$$

identifying T and the interval [0, 1].

These neighbourhoods $\mathcal{M}(q, a)$ will be referred to as major arcs. There is $\nu_1 > 0$ such that

$$|f(t)| < CN^{1-\nu_1}, \tag{4.6}$$

if $t$ does not belong to a major arc;

$$f(t) = \frac{1}{q} S(q, a) v(t - a/q) + O(N^{1/2}) \quad \text{if } t \in \mathcal{M}(q, a), \tag{4.7}$$

where

$$S(q,a)=\sum_{1}^{q} e^{2\pi i r^2 a/q}. \tag{4.8}$$

$$v(\beta)=N\int_0^1 e^{2\pi i N^2\beta u^2}du. \tag{4.9}$$

The reader will find details on these matters in [Vaug] for instance. For our purpose here, only the estimate

$$|f(t)| \leqq Cq^{-1/2}(|t-a/q|+N^{-2})^{-1/2} \quad \text{for } t\in\mathcal{M}(q,a) \tag{4.10}$$

will be relevant.

The only difficulty in deriving (4.2) from (4.6), (4.10) is to pass from (4.4), i.e. coefficients 1, to a general coefficiental sequence $\bar{a}=(a_n)_{1\leqq n\leqq N}$, $|\bar{a}|=1$. we assume $\delta>1/\alpha(N)$.

Let $t_1,\dots,t_R$ be $1/N^2$-separated points in [0, 1] satisfying

$$\left|\sum_{1}^{N} a_n e^{2\pi i n^2 t_r}\right| > \delta N^{1/2} \qquad (1\leqq r\leqq R). \tag{4.11}$$

Our purpose is to estimate $R$. By linearization, (4.11) yields

$$\sum_{1\leqq r,r'\leqq R}\left|\sum_{1}^{N}\exp[2\pi i n^2(t_r-t_{r'})]\right| > \delta^2 NR^2. \tag{4.12}$$

Letting $\gamma>2$ be fixed, (4.12) and Hölder's inequality imply

$$\sum_{1\leqq r,r'\leqq R}|f(t_r-t_{r'})|^{\gamma} > \delta^{2\gamma}N^{\gamma}R^2. \tag{4.13}$$

Consider the function

$$F(\theta)=(N^2|\sin\theta|+1)^{-\gamma/2} \tag{4.14}$$

satisfying

$$\| F\|_1 \sim 1/N^2. \tag{4.15a}$$

Define

$$G(t)=\sum_{\substack{q\leqq Q\\ 0\leqq a<q}} q^{-\gamma/2}F(t-a/q) \qquad \text{where } Q\sim\delta^{-5}. \tag{4.15b}$$

Thus (4.13) implies

$$\sum_{1\leqq r,r'\leqq R} G(t_r-t_{r'})>\delta^{2\gamma}R^2. \tag{4.16}$$

Let $0 \leqq \sigma \leqq C$ be a function on $\mathbf{T}$ verifying the conditions

$$\operatorname{supp} \hat{\sigma} \subset [-N^2, N^2], \tag{4.17}$$

$$\| \sigma \|_1 < CR/N^2, \tag{4.18}$$

$$\sigma \geqq 1 \text{ on a } 1/N^2\text{-neighbourhood of } \{t_1, \ldots, t_R\}. \tag{4.19}$$

Denote $\mathscr{K}$ the indicator function of a $1/10N^2$-neighbourhood of $0 \in \mathbf{T}$. By the separation-hypothesis of the points $t_r$ and (4.19)

$$\sigma \geqq \sum_1^R \mathscr{K}(t - t_r). \tag{4.20}$$

Define $\sigma_1$ by $\sigma_1(\theta) = \sigma(-\theta)$. Thus, from (4.20)

$$(\sigma * \sigma_1)(t) \geqq C \sum_{1 \leqq r, r' \leqq R} N^{-2} \mathscr{K}(t - (t_r - t_{r'})). \tag{4.21}$$

Since $F(\theta) \sim F(\theta')$ for $|\theta - \theta'| < 1/N^2$, it follows from (4.16), (4.21) that

$$\langle G, \sigma * \sigma_1 \rangle \geqq C N^{-4} \delta^{2\gamma} R^2. \tag{4.22}$$

Expressing the left member of (4.22) in Fourier coefficients, one gets

$$\sum_{|k| \leqq N^2} |\hat{\sigma}(k)|^2 |\hat{G}(k)| \geqq C N^{-4} \delta^{2\gamma} R^2, \tag{4.23}$$

and we estimate the left member.

It follows from (4.15b) that

$$\hat{G}(k) = \sum_{\substack{q \leqq Q \\ q | k}} q^{1-\gamma/2} \hat{F}(k). \tag{4.24}$$

Hence, by (4.15a)

$$|\hat{G}(k)| \leqq C N^{-2} d(k; Q), \tag{4.25}$$

where $d(k; Q)$ stands for the number of divisors of $k$ less than $Q$.

Fix $\tau > 0$. Estimate by (4.18), (4.25) and the construction of $\sigma$

$$\begin{aligned} &\sum_{|k| \leqq N^2} |\hat{\sigma}(k)|^2 |\hat{G}(k)| \\ &\leqq C N^{-2} \sum_{|k| \leqq N^2} d(k; Q) |\hat{\sigma}(k)|^2 \\ &\leqq C N^{-4} Q^\tau R + C Q R^2 N^{-6} |\{0 \leqq k \leqq N^2 \mid d(k; Q) > Q^\tau\}|. \end{aligned} \tag{4.26}$$

Thus

$$CQ^{\tau} + CQRN^{-2}|\{0 \leqq k \leqq N^2 \mid d(k; Q) > Q^{\tau}\}| \geqq C\delta^{2\gamma}R. \tag{4.27}$$

LEMMA 4.28. *$|\{0 \leqq k \leqq N \mid d(k; Q) > Q^{\tau}\}| < C_{\tau,B}Q^{-B}N$ whenever $0 < \tau < B < \infty$.*

Once the lemma is proved, (4.27) yields $R \leqq C\delta^{-2\gamma-5\tau} < \delta^{-4-\varepsilon}$, since $\gamma > 2$, $\tau > 0$ are arbitrary. This is the desired inequality (4.2).

PROOF OF LEMMA 4.28. For $2 \leqq q \leqq Q$ define the function $\mathcal{Y}_q$ on $[0, N]$, putting

$$\begin{aligned} \mathcal{Y}_q(k) &= 1 \qquad \text{if } q \mid k, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{4.29}$$

Fixing any integer $B \geqq 1$, write

$$|\{0 \leqq k \leqq N \mid d(k; Q) > Q^{\tau}\}| \leqq Q^{-\tau B} \sum_1^N \Big[\sum_{2 \leqq q \leqq Q} \mathcal{Y}_q(k)\Big]^B. \tag{4.30}$$

Denote $[q_1, q_2, \ldots, q_B]$ the smallest common multiple. Then, expanding the power, since $N$ is sufficiently large

$$\begin{aligned} &\frac{1}{N}\sum_1^N \Big[\sum_{q \leqq Q} \mathcal{Y}_q(k)\Big]^B \\ &\sim \sum_{q_1, q_2, \ldots, q_B \leqq Q} [q_1, q_2, \ldots, q_B]^{-1} \\ &\leqq \sum_{2 \leqq q \leqq Q^B} \frac{1}{q} d(q)^B < \exp\left(CB^2 \frac{\log Q}{\log\log Q}\right). \end{aligned} \tag{4.31}$$

Substituting (4.31) in (4.30), one has

$$\frac{1}{N}|\{0 \leqq k \leqq N \mid d(k; Q) > Q^{\tau}\}| \leqq Q^{-\tau B/2}, \tag{4.32}$$

for $Q$ large enough. Since $B$ was chosen arbitrarily this proves the lemma. ∎

The proof of Theorem 1 is now complete.

We indicate the modifications of previous argument in order to obtain Theorems 2 and 3.

Fixing a power $k$, it has to be shown that

(4.33) $$K_p(\{n^k \mid 1 \leqq n \leqq N\}) \leqq CN^{1/2-k/p},$$

for sufficiently large $p$, dependent on $k$. We first show how to substitute (1.9) in order to restrict ourselves to large values of $\delta$. Let $\bar{a} = (a_n)_{1 \leqq n \leqq N}$, $|\bar{a}| = 1$ and denote

(4.34) $$\varphi(t) = \sum_1^N a_n e^{in^k t}.$$

Write

(4.35) $$\int_{\{|\varphi| < \delta_0 N^{1/2}\}} |\varphi(t)|^p dt \leqq (\delta_0 N^{1/2})^{p-2} < N^{p/2-k},$$

provided

(4.36) $$\delta_0 < N^{(1-k)/(p-2)}.$$

Letting $p \to \infty$, one may thus restrict to values $\delta > N^{-\tau}$, with $\tau$ arbitrary small, in proving a distributional inequality

(4.37) $$\operatorname{mes}\{t \in \mathbf{T} \mid |\varphi(t)| > \delta N^{1/2}\} < C\delta^{-p+\varepsilon} N^{-k},$$

that will imply (4.33). Letting now

(4.38) $$f(t) = \sum_1^N e^{2\pi i n^k t},$$

the relevant behaviour of $f$ is taken care of by H. Weyl's inequality and the description of $f$ on major arcs (cf. [Vaug]). The rest of the above argument remains analogous. Here the exponent $\gamma$ has to be taken sufficiently large, again depending on $k$. The details are very much routine and we leave them to the reader.

In Theorem 3, the prime numbers $\{1 \leqq p \leqq N \mid p \text{ prime}\}$ are considered. We claim that for $r > 2$, one has

(4.39) $$K_r(\{1 < \text{primes} < N\}) \leqq C_r N_1^{1/2} N^{-1/r},$$

letting $N_1 = N/\log N$, i.e. the size of the set under consideration.

For $\varphi(t) = \Sigma_1^N a_p e^{2\pi i p t}$, $\Sigma |a_p|^2 \leqq 1$, write

(4.40) $$\int_{\{|\varphi| < \delta_0 N^{1/2}\}} |\varphi(t)|^r dt \leqq (\delta_0 N_1^{1/2})^{r-2} \leqq N_1^{r/2} N^{-1},$$

provided

$$\delta_0 < C(\log N)^{-1/(r-2)}. \tag{4.41}$$

Thus the distributional inequality (for any $\varepsilon > 0$)

$$\operatorname{mes}\{t \in \mathbf{T} \mid |\varphi(t)| > \delta N_1^{1/2}\} < C_\varepsilon \delta^{-2-\varepsilon} N^{-1} \tag{4.42}$$

only needs to be verified for $\delta > (\log N)^{-C}$.

The exponential sums with prime frequences are

$$f(t) = \sum_1^N \log p e^{2\pi i p t}. \tag{4.43}$$

They are introduced by writing

$$\begin{aligned}\varphi(t_r) &= \sum_1^N \frac{a_p}{\sqrt{\log p}} \sqrt{\log p}\, e^{2\pi i p t_r} \\ &= \sum_{N^{1/2}}^N \frac{a_p}{\sqrt{\log p}} \sqrt{\log p}\, e^{2\pi i p t_r} + O(N_1^{1/2}),\end{aligned} \tag{4.44}$$

and linearizing to get (4.12) above. Thus

$$\frac{1}{\log N} \sum_{1 \leqq r, r' \leqq R} \left| \sum_{N^{1/2}}^N \log p \exp[2\pi i p(t_r - t_{r'})] \right| > C\delta^2 N_1 R^2. \tag{4.45}$$

Hence

$$\sum_{1 \leqq r, r' \leqq R} |f(t_r - t_{r'})| > C\delta^2 N R^2 \tag{4.46}$$

or, fixing some $\gamma > 1$,

$$\sum_{1 \leqq r, r' \leqq R} |f(t_r - t_{r'})|^\gamma > C\delta^{2\gamma} N^\gamma R^2. \tag{4.47}$$

The relevant behaviour of $f$ is described in terms of major arcs

$$\mathcal{M}(q, a) = \{t \in \mathbf{T} \mid |t - a/q| \leqq PN^{-1}\}, \tag{4.48}$$

for $1 \leqq a \leqq q < P$, $(a, q) = 1$. Here $P$ stands for a sufficiently large power of $\log N$. The non-negligible contribution to $f$ when majorizing the left member of (4.47) come from these major arcs. Further, for $t \in \mathcal{M}(q, a)$

(4.49)
$$f(t)=\frac{\mu(q)}{\varphi(q)}v(t-a/q)+O(Ne^{-C\sqrt{\log N}}),$$

where $\mu$ is the Moebius function, $\varphi$ the number of Dirichlet characters to the modulus $q$ and

$$v(\beta)=\sum_1^N e^{2\pi in\beta}\quad \text{(see [Vaug], section 3, for details).}$$

We denote

(4.50)
$$F(\theta)=(N|\sin\theta|+1)^{-\gamma},$$

(4.51)
$$G(t)=\sum_{\substack{q\leqq Q\\ 0\leqq a<q}}\varphi(q)^{-\gamma}F(t-a/q).$$

Then $\|F\|_1\sim 1/N$ and (4.47) implies

$$\sum_{1\leqq r,r'\leqq R}G(t_r-t_{r'})>C\delta^{2\gamma}R^2.$$

Using the fact that $\varphi(q)>C_\tau q^{1-\tau}$ for any $\tau>0$, one proceeds as in the case of the squares, discussed above, to bound

(4.52)
$$\sum_{1\leqq r,r'\leqq R}G(t_r-t_{r'})\leqq C_\varepsilon R^{1+\varepsilon}.$$

Hence $R<C_\varepsilon\delta^{-2-\varepsilon}$ and (4.42).

Remarks. (1) There is an obvious extension of Theorem 2 to polynomial sequences $\{\Psi(n)\mid n=1,2,\dots\}$ where $\Psi(x)$ is a given polynomial (with integer coefficients), using the same methods as above.

(2) We only discussed $\Lambda_p$-sets in the context of subsets of **Z**, i.e. characters on **T**. Considering lattice points on the sphere,

$$\left\{(m_1,m_2,\dots,m_d)\in\mathbf{Z}^d\;\middle|\;\sum_1^d m_j^2=N^2\right\},$$

for large dimension $d\geqq 5$, one may get $\Lambda_p$-subsets in $\{1,2,\dots,N\}^d$ of maximal size, i.e. $N^{2d/p}$, for suitable $p>p(d)$. One has in particular that $p(d)\to 2$ for $d\to\infty$. Lattice point sets on spheres have sometimes been considered in the context of this problem.

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